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Dissipation and Large Thermodynamic Fluctuations

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The results of recent work of Kipnis, Olla, and Varadhan on the dynamic large deviations from a hydrodynamic limit for some interacting particle models are formally extended to a general hydrodynamic situation, including nonequilibrium steady states, as a fluctuation-dissipation hypothesis. The basic conjecture is that the exponent of decay in the probability of a large thermodynamic fluctuation is given by the dissipation of the force required to produce the fluctuation. It is shown that this hypothesis leads to a nonlinear version of Onsager–Machlup fluctuation theory that had previously been proposed by Graham. A direct consequence of the theory is a dynamic variational principle for the most probable thermodynamic history subject to imposed constraints (Onsager's principle of least dissipation). Following Graham, the theory leads also to a generalized potential, analogous to an equilibrium free energy, for the nonequilibrium steady state and an associated static variational principle. Finally, a formulation of nonlinear fluctuating hydrodynamics is proposed in which the noise enters multiplicatively so as to reproduce the conjectured large-deviations theory on a formal analogy with the results of Freidlin and Wentzell for finite-dimensional systems.

KEY WORDS: Fluctuations; thermodynamics; fluctuation-dissipation relation; nonlinear hydrodynamics; large deviations; generalized thermodynamic potentials.

1. INTRODUCTION

The famous formula of Boltzmann, $S = k_B \log W$, defined the macroscopic entropy of the second law in terms of microscopic probabilities.[1] It was Einstein in 1907[2,3] who had the ingenious idea to invert this relation to allow the calculation of microscopic probabilities from thermodynamics:

$$W \sim \exp \frac{S}{k_B} \tag{1}$$

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The latter "Einstein fluctuation formula" is now the basis of the standard theory for static equilibrium fluctuations.[4–6]

Inspired by this formula, Onsager and Machlup in 1953 gave a very suggestive reformulation of linear fluctuation theory about equilibrium.[7] The starting point of their work was a linear Langevin equation for the thermodynamic fluctuation variables $\alpha = (\alpha_1, ..., \alpha_n)$ which was presumed to be obtained from microscopic theory by virtue of a kind of central limit theorem. From this they obtained a formula for multitime probability density functions of the fluctuation variables in the form

$$f_p\begin{pmatrix} \alpha^{(1)} \cdots \alpha^{(p)} \\ t_1 \cdots t_p \end{pmatrix} \propto \exp\left[-\min \int_{-\infty}^{+\infty} dt\, L(\dot{\alpha}, \alpha) \right] \tag{2}$$

where "min" denotes minimization subject to the constraints $\alpha(t_1) = \alpha^{(1)}, ..., \alpha(t_p) = \alpha^{(p)}$, and $L(\dot{\alpha}, \alpha)$ is a "Lagrangian" given by

$$L(\dot{\alpha}, \alpha) = \frac{1}{2k_{\mathrm{B}}} \left[\Phi(\dot{\alpha}, \dot{\alpha}) + \Psi(\mathbf{X}, \mathbf{X}) - \frac{d}{dt} S(\alpha) \right] \tag{3}$$

We explain briefly the notation. S is the entropy, which, written in terms of the fluctuations variables α, is assumed to be given as a quadratic form:

$$S = S_0 - \frac{1}{2} \sum_{ij} s_{ij} \alpha_i \alpha_j \tag{4}$$

$\mathbf{X}$ is the thermodynamic force defined as a function of α by

$$X_i = \frac{\partial S}{\partial \alpha_i} \tag{5}$$

The functions Φ and Ψ are also positive-definite quadratic forms (dissipation functions) given by

$$\Phi(\dot{\alpha}, \dot{\alpha}) = \frac{1}{2} \sum_{ij} R_{ij} \dot{\alpha}_i \dot{\alpha}_j \tag{6}$$

and

$$\Psi(\mathbf{X}, \mathbf{X}) = \frac{1}{2} \sum_{ij} L_{ij} X_i X_j \tag{7}$$

where L_{ij} are the Onsager coefficients in the linear phenomenological laws:

$$\dot{\alpha}_i = \sum L_{ij} X_j \tag{8}$$

and $R_{ij} = (L^{-1})_{ij}$. Onsager and Machlup assessed their own achievement as follows: "The theorem which has been proved is seen to be analogous to the Boltzmann principle. The latter tells the probability of a state in terms of its entropy; this theorem tells the probability of a temporal succession of states in terms of the entropy and dissipation function." The Onsager–Machlup result (2), in fact, is shown without difficulty to lead to

$$f_1\begin{pmatrix}\alpha\\0\end{pmatrix} \propto \exp\left[-\frac{1}{2k_\mathrm{B}}\sum_{ij} s_{ij}\alpha_i\alpha_j\right] \tag{9}$$

i.e., to give the Einstein fluctuation formula (in quadratic approximation.) What is more, the formula provides a dynamic variational principle (principle of least dissipation) for the most probable histories, which generalizes the familiar equilibrium variational principles (principle of maximum entropy, etc.) for the most probable states. The extension of the approach to other situations where a linear fluctuation theory applies, such as the steady state of open systems,[8] is, as observed already by Onsager and Machlup, straightforward.

However, the extension of the Onsager–Machlup approach to the case of large (or nonlinear) fluctuations has not been so obvious or universally agreed upon. A noteworthy proposal for such a theory in a general hydrodynamic context was made by Graham at the XVIIth International Solvay Conference in Physics in 1978 (see ref. 9 and references therein.) Graham's starting point was an essentially phenomenological one, based on nonlinear fluctuating hydrodynamics. That is, white noise fluctuating currents were directly incorporated into the full *nonlinear* hydrodynamic equations. The microscopic basis for these equations is not as obvious as for the linear version, which presumably rests upon a central limit theorem, and the equations have been subject to criticism on several grounds.[10,11] Nevertheless, invoking certain asymptotic methods of evaluation, Graham argued for his results. Although his conclusions were certainly attractive, the method of argument could not be considered compelling.

Recently, a remarkable theorem has been proved which sheds light upon the issue. In a particular simple stochastic model of hydrodynamics, so-called symmetric simple exclusion dynamics, Kipnis, Olla and Varadhan (KOV) have obtained an exact asymptotic formula for the probability of hydrodynamic histories which differ by an arbitrarily large amount from the (overwhelmingly) most probable history calculated from the hydrodynamic equation[12] (see also ref. 13). The result is a rigorous one obtained by the probabilistic method of large deviations from the *hydrodynamic scaling limit* for the microscopic model. In the latter approach, the hydrodynamic equation is derived in a limit as a separation of scales parameter goes to zero. It was observed subsequently by Spohn that the result of KOV has a simple thermodynamic interpretation: the probability of a large fluctuation is given by the exponent of minus the dissipation required to produce the fluctuation.[14] This recalls the result in Eq. (2) of Onsager and Machlup. In fact, the theorem of KOV fully confirms the nonlinear Onsager–Machlup theory of Graham for their particular model and, furthermore, provides it a fundamental microscopic basis. Because the models to which the argument of KOV applies seem to have quite typical statistical mechanical behavior in other respects, the result strongly suggests the validity of the theory for real physical systems encountered in the laboratory.

We give in outline the plan of this paper. In Section 2, the standard Einstein theory of static equilibrium fluctuations is shown to have a rigorous mathematical basis in the existence of the thermodynamic limit of equilibrium partition functions. This example serves as a prototype of a large-deviations theory. We believe that the language and basic concepts of large deviations should be more widely known by statistical physicists, as they naturally apply in many situations. In Section 3, the rigorous result of Kipnis, Olla, and Varadhan is reviewed. The notion of a hydrodynamic scaling limit and the stochastic lattice gas models which are the subject of the theorem are both briefly introduced. Then, a precise mathematical statement, without proof, is given of the theorem of KOV. In Section 4, the final result of KOV is formally generalized as a fluctuation-dissipation hypothesis to a general hydrodynamic situation, including the nonequilibrium steady state. It is shown that this hypothesis leads to the nonlinear hydrodynamic fluctuation theory of Onsager–Machlup–Graham, which is briefly reviewed. A particular consequence discussed is a dynamical variational principle for the most probable hydrodynamic history subject to arbitrarily large constraints (a nonlinear version of Onsager's principle of least dissipation.) In Section 5, a general principle of large-deviations theory, the contraction principle, is introduced and formally applied to yield a generalized potential, analogous to an equilibrium free energy, for the nonequilibrium steady state. Under certain assumptions of time-reversal invariance, the generalized potential is explicitly calculated. Associated to the potential is a variational principle, which is a static version of Onsager's principle of least dissipation. In the final Section 6, our theory is compared with a more traditional theory of nonlinear hydrodynamic fluctuations based on stochastic differential equations à la Landau–Lifshitz. The large-deviations approach is shown to avoid the difficulties of a nonlinear version of the latter which were pointed out by van Saarloos *et al.*[11] and by van Kampen.[10] It is argued that the nonlinear hydrodynamic fluctuation theory, in contrast to the linear case, must explicitly contain a small parameter (here, the ratio of scales) for a precise interpretation which is free of paradoxes. Such a formulation is proposed in which the white noise enters multiplicatively so as to reproduce the conjectured large-deviations theory on a formal analogy with the results of Freidlin and Wentzell[15,16] for finite-dimensional systems.

2. EINSTEIN–BOLTZMANN FLUCTUATION THEORY AND LARGE DEVIATIONS

It was not until the rigorous proofs of the existence of the thermodynamic limit were given in the 1960s[17,18] that the Einstein fluctuation formulas such as Eq. (1) were given a sound mathematical basis. The connection between the thermodynamic limit and fluctuation theory is described lucidly and at length in the work of Martin-Löf.[19] Here we shall be brief.

Let $\Omega'_\Delta(E, N, \Lambda)$ denote the partition function of the microcanonical distribution on the energy shell of width Δ about the mean value E for N particles in the spatial domain Λ, i.e.,

$$\Omega'_\Delta(E, N, \Lambda) = \int_{q \in \Lambda^N, E \leqslant H(q,p) \leqslant E+\Delta} \frac{dp\,dq}{N!} \tag{10}$$

If the entropy at finite volume $S_\Delta(E, N, \Lambda)$, is defined as

$$S_\Delta(E, N, \Lambda) = k_B \log \Omega'_\Delta(E, N, \Lambda) \tag{11}$$

then the proof of the thermodynamic limit provides the existence, with $e = E/|\Lambda|$ and $n = N/|\Lambda|$, of

$$\lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} S_\Delta(E, N, \Lambda) = s_\Delta(e, n) \tag{12}$$

that is, the thermodynamic entropy per unit volume. Now, consider any "macroscopic variable" U. To be more precise, let U be an "m-body variable" of the form

$$U(x) = \sum_{\{i_1,\ldots,i_m\} \subset \{1,\ldots,N\}} u(x_{i_1},\ldots,x_{i_m}) \tag{13}$$

with $x_i = (q_i, p_i)$, $x = (q, p)$, and $u(x_1,\ldots,x_m)$ a symmetric function of its m variables, translation invariant, i.e.,

$$u(q_1 + a, p_1;\ldots; q_m + a, p_m) = u(q_1, p_1;\ldots; q_m, p_m)$$

and, for simplicity, finite range [so that $u(x_1,..., x_m)=0$ if $|q_i-q_j|\geqslant R$ for any i, j]. Then, for any "nice" set $A\subseteq \mathbf{R}$,

$$P_{\Lambda,E,N,\Delta}\left(\frac{U}{|\Lambda|}\in A\right)=\frac{1}{\Omega'_\Lambda(E,N,\Delta)}\int_{q\in\Lambda^N,E\leqslant H(x)\leqslant E+\Delta,U/|\Lambda|\in A}\frac{dx}{N!} \tag{14}$$

gives the probability that $U/|\Lambda|\in A$ for the microcanonical distribution. The same argument applied to establish the limit in Eq. (12) gives here the result that

$$\lim_{|\Lambda|\to\infty}\frac{1}{|\Lambda|}\log P_{\Lambda,E,N,\Delta}\left(\frac{U}{|\Lambda|}\in A\right)=\frac{s_\Delta(A\mid e,n)-s_\Delta(e,n)}{k_\mathrm{B}} \tag{15}$$

The quantity $s_\Delta(A\mid e,n)$ is often referred to in the physics literature as a "conditional entropy," i.e., it is the thermodynamic entropy subject to the condition $u\in A$ (see ref. 5, Chapter 6). In fact,

$$s_\Delta(A\mid e,n)=\sup_{u\in A}s_\Delta(u\mid e,n) \tag{16}$$

where $s_\Delta(u\mid e,n)=\inf_{A\ni u}s_\Delta(A\mid e,n)$ is an extended entropy function, concave and upper semicontinuous as a function of u. It is clear from the way it was defined that $\Delta s_\Delta(A\mid e,n)=s_\Delta(A\mid e,n)-s_\Delta(e,n)\leqslant 0$. This inequality also corresponds to the fact that the entropy is maximum for equilibrium and decreases subject to any constraint such as $u\in A$. We may now employ a short-hand notation for the limit statement in Eq. (15):

$$P_{\Lambda,E,N,\Delta}\left(\frac{U}{|\Lambda|}\in A\right)\sim\exp\left[|\Lambda|\frac{\Delta s_\Delta(A\mid e,n)}{k_\mathrm{B}}\right] \tag{17}$$

Thus, we have established a familiar form of the Einstein fluctuation formula.[5] Other versions of the formula, e.g., with the entropy difference replaced by a negative free energy difference, are derived in the same manner starting from an appropriate canonical distribution.[19]

The above example is, in fact, the prototype of a situation that occurs sufficiently often in probability theory that its basic features have been abstracted (ref. 20, Section 3). Let $\mathscr{X}$ be a complete separable metric space (or Polish space), $\mathscr{B}(\mathscr{X})$ the Borel σ-field of $\mathscr{X}$, and $\{P_n\mid n=1,2,...\}$ a sequence of probability measures on $\mathscr{B}(\mathscr{X})$. We say that $\{P_n\}$ has a *large-deviations property* if there exists a sequence of positive numbers $\{a_n\mid n=1,2,...\}$ which tend to ∞ and a function I from $\mathscr{X}$ into $[0,\infty]$ such that the following hold:

1. $I(x)$ is lower semicontinuous.

2. $I(x)$ has compact level sets.
3. $\limsup_{n\to\infty} a_n^{-1} \log P_n\{K\} \leqslant -\inf_{x\in K} I(x)$ for each closed K in $\mathscr{X}$.
4. $\liminf_{n\to\infty} a_n^{-1} \log P_n\{G\} \geqslant -\inf_{x\in G} I(x)$ for each open G in $\mathscr{X}$.

$I(x)$ is called a *rate function* of $\{P_n\}$. In the statistical mechanical example above, $I(u) = -k_B^{-1}(s_\Delta(u \mid e, n) - s_\Delta(e, n))$. The book of Ellis[21] develops large-deviations theory in detail and elaborates its many relations to equilibrium statistical mechanics. Here we shall simply make a few brief remarks. First, the rate function is unique. As direct consequences of the assumed properties 1–4 above, it follows that $I(x)$ attains its infimum over any nonempty closed subset of $\mathscr{X}$ and that the infimum over $\mathscr{X}$ itself is zero. A point of $\mathscr{X}$ where I vanishes is called a *minimum point*. If K is any closed subset of $\mathscr{X}$ not containing a minimum point, then there exists a number $I_0 > 0$ such that $P_n\{K\} \leqslant e^{-a_n I_0}$ for sufficiently large n. In particular, if there is a unique minimum point x_0, then the $\mathscr{X}$-valued random variables X_n on $\mathscr{X}$ with distributions P_n converge in probability to the deterministic limit x_0 at an exponential rate as $n \to \infty$. That is, the large-deviations property implies an exponential form of the law of large numbers. Finally, we remark that there is a large class of sets for which equality holds in properties 3–4 above. Call a Borel subset A of $\mathscr{X}$ an I-continuity set if

$$\inf_{x\in \mathrm{cl}\, A} I(x) = \inf_{x\in \mathrm{int}\, A} I(x) \tag{18}$$

If A is an I-continuity set, then $\lim_{n\to\infty} a_n^{-1} \log P_n\{A\} = -\inf_{x\in A} I(x)$.

3. LARGE DEVIATIONS FROM THE HYDRODYNAMIC LIMIT

The hydrodynamic description of a fluid represents a vast simplification of the detailed microscopic picture of motion of the order of 10^{23} molecules described by Newton's laws. In the reduced hydrodynamic picture, only the few locally conserved variables are retained. It is a common wisdom that this reduction is possible because of the presence of two well-separated characteristic length scales in the fluid: one, denoted by l, is the mean free path of a molecule, while the other is the length scale L over which the hydrodynamic variables vary (see, e.g., ref. 22; however, the idea is quite old, appearing, for instance, in the Hilbert or Chapman–Enskog solutions of the Boltzmann equation). For a typical fluid, the ratio $\varepsilon = l/L \sim 10^{-5}$ or smaller. For simple stochastic lattice gas models of hydrodynamic behavior, the above idea may be formalized in terms of a *hydrodynamic scaling limit*.[23] In such a context, hydrodynamic equations describe the evolution of the conserved quantities with probability approaching unity (law of large numbers) in the idealized limit as $\varepsilon \to 0$. We shall here first introduce a simple class of lattice gas models and then describe in precise terms the hydrodynamic scaling limit for these models. For more details see refs. 14, 23 and 24.

The models we consider live on a discrete lattice. Anticipating the hydrodynamic scaling, we take as our lattice a subset of the hypercubic lattice $\varepsilon \mathbf{Z}^d$ with lattice spacing ε, consisting of all the lattice sites in the "macroscopic region" Λ, i.e., $\Lambda_\varepsilon = \varepsilon \mathbf{Z}^d \cap \Lambda$. Here, Λ is a simply-connected domain in $\mathbf{R}^d$ with smooth boundary. Particles jump stochastically from site to site of this lattice subject to an exclusion condition of a single particle per site. We shall restrict attention to the case where the particle may jump only to lattice sites a fixed distance R (in microscopic units ε) from its own position. Additionally, the hopping rate of a particle may depend on the occupancies of the neighboring sites within the finite range R of its position. Nondegeneracy is presumed, in that the exchange rate is strictly positive for a pair of occupied and unoccupied sites within a distance R. Note that in this dynamics, often referred to as a *Kawasaki dynamics*, particles are neither created nor destroyed, and total particle number is the only conserved quantity. In more mathematical terms, a microscopic state of the model is given by a vector $\eta \in \Omega_\varepsilon = \{0, 1\}^{\Lambda_\varepsilon}$, whose components give the occupation numbers of the lattice sites. A probability distribution μ over the states evolves according to a "master equation":

$$\frac{d}{dt} \mu_t(\eta) = \frac{1}{2} \sum_{x, y \in \Lambda_\varepsilon} [c(x, y; \eta^{x,y}) \mu_t(\eta^{x,y}) - c(x, y; \eta) \mu_t(\eta)] \tag{19}$$

In this equation, $c(x, y; \eta)$ is the probability rate per unit time for an exchange between lattice sites x and y when the lattice configuration is given by η. By $\eta^{x,y}$ we denote the configuration obtained from η by exchanging occupancies at sites x and y. The rate $c(x, y; \eta)$ is subject to the conditions stated previously. Furthermore, we assume that the dynamics in the bulk of the domain Λ_ε are translation invariant, i.e.,

$$c(x + a, y + a; \tau_a \eta) = c(x, y; \eta) \tag{20}$$

when the distances of all of x, y, $x + a$, and $y + a$ from $\partial \Lambda$ are greater than R. We have denoted by $\tau_a \eta$ a translated configuration, $\tau_a \eta(x) = \eta(x - a)$. Also, we assume the rates satisfy a condition of *detailed balance* with respect to a translation-invariant, finite-range Hamiltonian $H(\eta)$, i.e.,

$$c(x, y; \eta) \exp[-\beta H(\eta)] = c(x, y; \eta^{x,y}) \exp[-\beta H(\eta^{x,y})] \tag{21}$$

With this assumption, the unique invariant measure under the dynamics of

Eq. (19) for a fixed total number of particles N is the canonical Gibbs measure

$$\mu_{\text{eq}}(\eta) = Z_{\text{eq}}^{-1} \exp[-\beta H(\eta)]\, \delta_{N(\eta), N} \tag{22}$$

On a macroscopic scale, the locally conserved particle number density $\rho(q, \tau)$, as a function of macroscopic position $q \in \Lambda$ and macroscopic time $\tau \geqslant 0$, is expected to obey a nonlinear diffusion equation

$$\partial_\tau \rho(q, \tau) = \partial_q \cdot [D(\rho(q, \tau)) \cdot \partial_q \rho(q, \tau)] \tag{23}$$

In this equation, the *bulk diffusion matrix* $D(\rho)$ is expected to be given by a Green–Kubo formula as the time integral of an equilibrium current–current correlation function at the density ρ. In fact, for many cases these expectations may be confirmed in the following precise sense. To keep things simple, let us take $\Lambda = [0, 1]^d$ with periodic boundary conditions. As initial measure for the dynamics, consider a *local equilibrium distribution*

$$\mu_{le}(\eta) = Z_{le}^{-1} \exp\left[-\beta H(\eta) + \sum_{x \in \Lambda_\varepsilon} \lambda_x \eta(x) \right] \tag{24}$$

determined by a smooth chemical potential profile $\lambda_0(q)$ on the macroscopic scale as $\lambda_x = \lambda_0(\varepsilon x)$. Observe that the initial profile varies slowly (order ε) on the microscopic scale, according to our earlier proposed requirement for hydrodynamic behavior. Now consider the empirical density field defined by

$$X_\tau^\varepsilon(\phi) = \varepsilon^d \sum_{x \in \Lambda_\varepsilon} \phi(\varepsilon x)\, \eta_{\varepsilon^{-2}\tau}(x) \tag{25}$$

where η_t gives the configuration evolved to (microscopic) time t. The smearing with respect to the test function ϕ is a smooth version of averaging over a macroscopic region $\Delta_\varepsilon = \varepsilon \mathbf{Z}^d \cap \Delta$, since only for a "coarse-grained density" can the hydrodynamic equations be expected to hold. Notice also that, whereas for positions, $x \sim \varepsilon^{-1} q$, for times, $t \sim \varepsilon^{-2}\tau$. This corresponds to the fact that, for a purely diffusive system, effects propagate over a lattice distance $\sim \varepsilon^{-1}$ on a time scale $\sim \varepsilon^{-2}$ and also to the symmetry of the macroscopic equation under the scalings $q \to \lambda q$, $\tau \to \lambda^2 \tau$. Let $\mathscr{M}_1$ denote the set of measurable functions from Λ into $[0, 1]$ with the weak topology, considered as a space of density profiles. By $D([0, T]; \mathscr{M}_1)$ denote the path space of histories of density profiles over the macroscopic time interval $[0, T]$ and by P^ε the probability measure on path space which is induced by the stochastic evolution of Eq. (19) with initial measure as in Eq. (24). Then, the following has been proved for a class of models[24,14]:

Theorem 1. (See refs. 14 and 24.) For each fixed $\phi \in C^{\infty}(\Lambda)$, $\tau \geqslant 0$, and $\delta > 0$,

$$\lim_{\varepsilon \to 0} P^{\varepsilon}\left(\left|X_{\tau}^{\varepsilon}(\phi) - \int_{\Lambda} \phi(q)\, \rho(q, \tau)\, dq\right| > \delta\right) = 0 \tag{26}$$

where $\rho(q, \tau)$ is the (weak) solution of the macroscopic equation (23) with initial condition $\rho_0(q)$ and, for each q, $\rho_0(q)$ corresponds to $\lambda_0(q)$ by the bulk equilibrium relation $\rho(\lambda)$ between density and chemical potential.

Thus, the hydrodynamic equation describes the evolution of the density in the sense of a law of large numbers as $\varepsilon \to 0$. It is also verified that $D(\rho)$ is given by a Green–Kubo formula and therefore $D(\rho) \geqslant 0$.[14]

At present the proofs of this theorem are subject to certain technical restrictions which largely confine the results to the one-dimensional case. We want to emphasize, however, that despite the simplicity and special nature of the models that can be treated, their equilibrium and non-equilibrium statistical mechanical behavior seem to fully conform to the predictions of conventional theory. This is already partially evidenced by the above result. In addition, the validity of *(linear) fluctuating hydrodynamics* (see ref. 26; also ref. 4, Part 2, Section 88, and ref. 5, Section 11.12) for time-dependent fluctuations of the density field about equilibrium has been established as a functional central limit theorem for $\varepsilon \to 0$. This result has been obtained in about the same generality as the law of large numbers.[24,14] There is also a theory of linear fluctuating hydrodynamics for the *nonequilibrium steady state* (see ref. 8 for a review) whose validity for one special case, the symmetric simple exclusion model, has been obtained as a central limit theorem by Spohn.[27] The symmetric simple exclusion model is, perhaps, the simplest model of our type and corresponds to the choice

$$c(x, y; \eta) = [\eta(x) - \eta(y)]^2 \tag{27}$$

These rates satisfy the detailed balance condition (21) with $\beta H(\eta) = 0$, i.e., this dynamics corresponds to an infinite-temperature situation. In this model, the only interaction between distinct particles is via the exclusion condition. Nevertheless, the model already exhibits quite complex and realistic behavior. In the cited paper, Spohn studied the simple exclusion dynamics in a finite slab when coupled at its opposite ends to particle concentration reservoirs, modeled stochastically by particle creation–annihilation rates obeying detailed balance with respect to two different chemical potentials. In the unique steady state for this situation, a density gradient is set up across the slab and a particle number current flows from the reservoir of higher chemical potential into the lower. In particular, Spohn verified the predictions of linear fluctuating hydrodynamics for the steady-state density-fluctuation covariance, including the interesting feature of a long-ranged part with a power-law decay. In real fluids subject to a temperature gradient such long-range decay is also predicted and observed in light-scattering experiments (see references in ref. 8).

Besides the purely diffusive models, we want to mention briefly some other types of models where similar results are established. For one-dimensional *asymmetric* simple exclusion the Burgers equation is derived in an "Euler limit" where $x \sim \varepsilon^{-1}q$ and $t \sim \varepsilon^{-1}\tau$.[23] For stochastic versions of the HPP and FHP models[22] the actual Euler equations have been derived in the same scaling and also the incompressible Navier–Stokes equations have been derived.[28] The incompressible Navier–Stokes equations are not invariant under any nontrivial space-time scaling; however, they are invariant under a transformation in which velocities are also rescaled: $q \to \lambda^{-1}q$, $t \to \lambda^{-2}t$, $u \to \lambda u$. A corresponding scaling is employed in the cited derivation. The basic difficulty with the Navier–Stokes equations is the presence of several distinct time scales: the microscopic time scale ε^0 on which relaxation to local equilibrium occurs, the "Euler" time scale ε^{-1} on which density effects propagate as sound waves, and the "Navier–Stokes" time scale ε^{-2} on which diffusive propagation occurs. It remains an important problem to characterize precisely the Navier–Stokes correction to the Euler limit and to specify in what regime and in what sense the system is better described by these equations. In any case, despite all the stated limitations, we wish to emphasize again that these simple models, although extreme caricatures of actual physical fluids, nevertheless exhibit quite realistic hydrodynamic behavior.

We are now prepared to discuss the deep and beautiful work of Kipnis, Olla, and Varadhan (KOV) on the large deviations from the hydrodynamic limit for the symmetric simple exclusion model.[12] It is not our intention here to explain in detail the methods of their proof, for which we refer the reader to the original paper and to an illuminating, heuristic discussion of Spohn (ref. 14, Part II, Chapter 3.7), but only to expose the results and explain their physical significance. The model considered by KOV was the symmetric simple exclusion dynamics of Eq. (27) on the macroscopic domain $\Lambda = [0, 1]$ with periodic boundary conditions, i.e., on the unit circle S^1. Therefore, for fixed $\varepsilon = 1/N$, the microscopic model was defined on $S^1_N = \{i/N \mid i = 1,..., N\}$. For this model, Theorem 1 holds with the macroscopic equation

$$\partial_\tau \rho(q, \tau) = \Delta_q \rho(q, \tau) \tag{28}$$

i.e., Eq. (23) with $D(\rho)=1$. Let P_α^ε denote the measure on the path space $D([0, T], \mathscr{M}_1)$ induced by taking the (grand canonical) Gibbs measure with chemical potential $\lambda(\alpha)$ (here, a Bernoulli measure with constant density α) as initial measure for the stochastic dynamics. For simplicity, we have chosen the case where the initial density is uniform and P_α^ε corresponds to a stationary stochastic process. To state the results of KOV, we need to make certain definitions and introduce some notation. We shall employ the following short-hand:

$$\langle \rho, \phi\rangle = \int_{S^1} \rho(q)\,\phi(q)\,dq, \qquad \rho \in \mathscr{M}_1,\ \phi \in C(S^1) \tag{29}$$

As before, given any path for the microscopic dynamics, $\{\eta_t \mid t \in [0, \varepsilon^{-2}T]\} \in D([0, \varepsilon^{-2}T]; \Omega_\varepsilon)$, the empirical density is defined by

$$\rho^\varepsilon(q, \tau) = \eta_{\varepsilon^{-2}\tau}([\varepsilon^{-1}q]) \tag{30}$$

where $[x]$ denotes the integer part of x. Thus, $\rho^\varepsilon \in D([0, T]; \mathscr{M}_1)$. We define also a Hilbert space $H^1(\sigma)$ for each $\sigma \in D([0, T], \mathscr{M})$ where $\mathscr{M}$ is the space of bounded measurable functions on Λ into $\mathbf{R}$, as follows: consider in $C^{2,1}(S^1, [0, T])$ the equivalence relation $U \sim U'$ if $U(q, \tau) - U'(q, \tau)$ is a function only of τ. On the equivalence classes $C^{2,1}/\sim$ define the inner product

$$(U, U') = \int_0^T \langle \sigma_\tau, \partial_q U \cdot \partial_q U'(\cdot, \tau)\rangle\, d\tau \tag{31}$$

and $H^1(\sigma)$ as the completion of $C^{2,1}/\sim$ with respect to this scalar product. We alert the reader that the elements of H^1 shall have the physical significance of "external potentials." For each $\rho \in D([0, T], \mathscr{M}_1)$ define the linear functional on $C^{2,1}(S^1, [0, T])$:

$$\begin{aligned} l(\rho; U) = {} & \frac{1}{2}\langle \rho_0, U(\cdot, 0)\rangle - \frac{1}{2}\langle \rho_T, U(\cdot, T)\rangle \\ & + \frac{1}{2}\int_0^T d\tau\, \langle \rho_\tau, (\partial_\tau + \Delta_q)\, U(\cdot, \tau)\rangle \end{aligned} \tag{32}$$

Then introduce the following rate functionals, for $\rho_0 \in \mathscr{M}_1$,

$$\begin{aligned} f_\alpha[\rho_0] = \sup_{\phi_0, \phi_1 \in C(S^1)} & [\langle \rho_0, \phi_0\rangle + \langle 1 - \rho_0, \phi_1\rangle \\ & - \langle 1, \log(\alpha e^{\phi_0} + (1-\alpha)\, e^{\phi_1})\rangle] \end{aligned} \tag{33}$$

and, for $\rho \in D([0, T], \mathcal{M}_1)$,

$$I_D[\rho] = \sup_{U \in C^{2,1}} \left[l(\rho; U) - \frac{1}{2} \int_0^T d\tau \, \langle \sigma(\rho(\cdot, \tau)), (\partial_q U(\cdot, \tau))^2 \rangle \right] \tag{34}$$

where $\sigma(\rho) = \rho(1-\rho)$. Finally, let

$$I_\alpha[\rho] = f_\alpha(\rho_0) + I_D(\rho) \tag{35}$$

Then Kipnis, Olla, and Varadhan establish the following theorem:

Theorem 2.[12] The measure P_α^ε has the large-deviations property as $\varepsilon \to 0$ with rate functional $I_\alpha[\rho]$. In particular, for any I_α-continuous Borel subset $A \subset D([0, T], \mathcal{M}_1)$

$$\lim_{\varepsilon \to 0} \varepsilon \log P_\alpha^\varepsilon(\rho^\varepsilon \in A) = -\inf_{\rho \in A} I_\alpha[\rho] \tag{36}$$

In ref. 14 it is explained how this result extends also to the same class of models for which Theorem 1 was proved, with appropriate definitions of the basic quantities. We now explain in detail the physical significance of this result.

The rate function which appears in the above theorem has two parts, a "static" contribution $f_\alpha[\rho_0]$ which involves only the time-zero profile ρ_0 and the "dynamic" part $I_D[\rho]$. It is not difficult to show that, if $f_\alpha(\rho) < \infty$, then

$$f_\alpha[\rho_0] = \left\langle \rho_0, \log\left(\frac{\rho_0}{\alpha}\right) \right\rangle + \left\langle 1 - \rho_0, \log\left(\frac{1-\rho_0}{1-\alpha}\right) \right\rangle \tag{37}$$

Furthermore, it is a simple computation that the conditional free energy, in the state with chemical potential $\lambda(\alpha)$, to have a density ρ, i.e.,

$$f(\rho \mid \lambda(\alpha)) = -s(\rho) - \lambda(\alpha)\rho + \Psi(\alpha) \tag{38}$$

[$\Psi(\alpha)$ normalizes f to zero for $\rho = \alpha$] is here given by

$$f(\rho \mid \lambda(\alpha)) = \rho \log \frac{\rho}{\alpha} + (1-\rho) \log \frac{1-\rho}{1-\alpha} \tag{39}$$

Thus,

$$f_\alpha[\rho_0] = \int_{S^1} f(\rho_0(q) \mid \lambda(\alpha)) \, dq \tag{40}$$

This is another familiar form of the Einstein fluctuation formula, for the hydrodynamic situation (e.g., see ref. 11). This result is not at all surprising, since it just states that, to observe a profile ρ, there is a cost in free energy to observe the initial fluctuation ρ_0 at time zero. Let us note that establishment of this form of the Einstein formula for static hydrodynamic fluctuations from equilibrium presents no great problem, even for rather realistic models, since it requires only good control over partition functions of local equilibrium distributions as $\varepsilon \to 0$ (for which, see ref. 29).

The real contribution of KOV lies in their calculation of the "dynamic" part $I_D[\rho]$ of the rate function, which for realistic systems lies far beyond our present mathematical abilities. To understand its significance, however, let us cite Lemma 5.1 of KOV.

Lemma 1.[12] If $I_D[\rho] < \infty$, there exists a $U \in H^1(\sigma(\rho))$ such that

$$I_D[\rho] = \frac{1}{4} \int_0^T d\tau \, \langle \sigma(\rho_\tau), (\partial_q U(\cdot, \tau))^2 \rangle \tag{41}$$

and that ρ satisfies in the weak sense the equation

$$\partial_\tau \rho(q, \tau) = \partial_q [\sigma(\rho(q, \tau)) \, \partial_q U(q, \tau) + \partial_q \rho(q, \tau)] \tag{42}$$

Now, by the Einstein relation $\sigma = \chi D$, $\sigma(\rho) = \rho(1-\rho)$ is the conductivity for the model. Therefore, Eq. (42) gives the evolution of the density in the presence of an external field $F(q, \tau)$ arising from a potential $U(q, \tau)$:

$$F(q, \tau) = -\partial_q U(q, \tau) \tag{43}$$

In fact, the basic strategy of the proof of KOV is to modify the symmetric simple exclusion dynamics by "turning on" a slowly varying external potential so that a given profile ρ becomes typical for the new dynamics. The change in the path measure for the new dynamics can be explicitly evaluated (by the Girsanov formula) and is given asymptotically by $\exp(-\varepsilon^{-1} I_D[\rho])$ as $\varepsilon \to 0$. The above lemma establishes that for any given ρ there is a (unique up to $\sim$) $U = U[\rho]$ for which Eq. (42) is satisfied. In the general case discussed by Spohn[14] the corresponding lemma states that there is a U such that (in the weak sense)

$$\partial_{\tau\rho}(q, \tau) = \partial_q [\sigma(\rho(q, \tau)) \, \partial_q U(q, \tau) + D(\rho(q, \tau)) \, \partial_q \rho(q, \tau)] \tag{44}$$

is satisfied. Now, Spohn has also made the following important observation: the dynamic part $I_D[\rho]$ of the rate function, as it appears in Eq. (41) of the above lemma, is just one-half of the time integral of the dissipation by the external field required to produce the fluctuation ρ (Ohm's law).

Thus, just as for static fluctuations, the probability may be calculated from thermodynamic considerations. The simplicity of the final result strongly suggests a more general principle. That is what we explore in the following section.

4. A FLUCTUATION-DISSIPATION HYPOTHESIS AND THE NONLINEAR FLUCTUATION THEORY OF ONSAGER–MACHLUP–GRAHAM

We now wish to formulate a general conjecture, based on the results of Kipnis, Olla, and Varadhan exposed in the previous section. Our starting point is a set of hydrodynamic balance equations written in *Onsager form*[8,9]:

$$\dot{\rho}_\mu = -\partial_i j^R_{i\mu}(\rho) + \partial_i(L_{i\mu,j\nu}(\rho)\, X_{j\nu}(\rho)) \tag{45}$$

To economize on space, we employ a compact notation. Roman indices range over spatial directions 1, 2,..., d, Greek indices range over conserved quantities 1, 2,..., A, and repeated indices are summed over. We consider these equations, as above, in a simply-connected domain $\Lambda \subset \mathbf{R}^d$ with smooth boundary $\partial\Lambda$. The ρ_μ, $\mu = 1,..., A$ are the conserved (hydrodynamic) variables. The $j^R_{i\mu}$ are certain functionals of the ρ's, referred to as the *reversible currents*. $X_{j\nu}(\rho)$ is a *thermodynamic force* defined as

$$X_{j\nu}(\rho) = \partial_j \lambda_\nu(\rho) \tag{46}$$

where $\lambda_\nu(\rho)$ are the *conjugate thermodynamic variables*:

$$\lambda_\nu(\rho; q) = -\frac{\delta S}{\delta\rho^\nu(q)} \tag{47}$$

with S the *entropy*

$$S[\rho] = \int_\Lambda dq\, s(\rho(q)) \tag{48}$$

Here, $s(\rho)$ is given by the bulk thermodynamic relation. $L_{i\mu,j\nu}(\rho)$ is the *Onsager coefficient* relating the *irreversible current* $j^D_{i\mu}$ and the thermodynamic force $X_{j\nu}$:

$$j^D_{i\mu}(\rho) = -L_{i\mu,j\nu}(\rho)\, X_{j\nu}(\rho) \tag{49}$$

The total current is then a sum

$$j_{i\mu}(\rho) = j^R_{i\mu}(\rho) + j^D_{i\mu}(\rho) \tag{50}$$

If the densities ρ_μ have the parities ε_μ under *time-reversal* T,

$$T\colon\ \rho_\mu \to \tilde{\rho}_\mu = \varepsilon_\mu \rho_\mu \tag{51}$$

then we assume transformation properties

$$S[\tilde{\rho}] = S[\rho] \tag{52}$$

$$L_{i\mu, j\nu}[\tilde{\rho}] = \varepsilon_\mu \varepsilon_\nu L_{j\nu, i\mu}[\rho] \tag{53}$$

The latter is the *Onsager–Casimir reciprocal relation*, which, for example, would follow as a consequence of a Green–Kubo formula for the Onsager coefficient. Likewise, we assume that the matrix of Onsager coefficients, for each ρ, is positive definite, as would also follow from a Green–Kubo representation. It is often convenient to introduce a (density-dependent) diffusion operator

$$(\hat{A}_{\mu\nu}(\rho) f_\nu)(q) = \partial_i (L_{i\mu, j\nu}(\rho)\, \partial_j f_\nu)(q) \tag{54}$$

In terms of this operator, the hydrodynamic equations may be written as

$$\dot{\rho}_\mu = -\partial_i j^R_{i\mu}(\rho) + \hat{A}_{\mu\nu}(\rho)\, \lambda_\nu(\rho) \tag{55}$$

From its definition, $\hat{A}_{\mu\nu}$ is clearly negative definite.

We now introduce *driven hydrodynamic equations* in the form

$$\dot{\rho}_\mu = -\partial_i j^R_{i\mu}(\rho) + \partial_i [L_{i\mu, j\nu}(\rho)(F_{j\nu} + X_{j\nu}(\rho))] \tag{56}$$

where $F_{j\nu}$ are the *external driving forces.* We shall often make the assumption that these driving forces are given by *external potentials* U_ν as

$$F_{j\nu} = \partial_j U_\nu \tag{57}$$

which allows us to write Eq. (56) instead as

$$\dot{\rho}_\mu = -\partial_i j^R_{i\mu}(\rho) + \hat{A}_{\mu\nu}(\rho)(U_\nu + \lambda_\nu(\rho)) \tag{58}$$

The boundary conditions we shall impose on the equations (45) or (55) shall generally be of the form

$$-\left.\frac{\delta S}{\delta \rho^\mu(q)}\right|_{\partial \Lambda} = \lambda^0_\mu(q) \tag{59}$$

with λ^0_μ a specified smooth function on $\partial \Lambda$. The *equilibrium* cases correspond to λ^0_μ constants, and the *nonequilibrium* situation to general boundary data.

The framework above is sufficiently general to incorporate almost every hydrodynamic situation of interest. Let us consider a few examples.

Examples

1. Purely Diffusive Systems. An A-component, purely diffusive system is governed by the nonlinear diffusion equation

$$\dot{\rho}_\mu = \partial_i [D_{\mu\nu}(\rho)\, \partial_i \rho_\nu(q)] \tag{60}$$

This may be written in the form of (1) with $j^R = 0$, $L_{i\mu, j\nu} = \delta_{ij} L_{\mu\nu}$,

$$L_{\mu\nu}(\rho) = D_{\mu\sigma}(\rho)\, \chi_{\sigma\nu}(\rho) \tag{61}$$

and

$$\chi_{\sigma\nu}(\rho) = \partial\rho_\sigma(\lambda)/\partial\lambda_\nu \tag{62}$$

i.e.,

$$\dot{\rho}_\mu = \partial_i [L_{\mu\nu}(\rho)\, \partial_i \lambda_\nu(q)] \tag{63}$$

Likewise, the driven case

$$\dot{\rho}_\mu = \partial_i [D_{\mu\nu}(\rho)\, \partial_i \rho_\nu(q) - L_{\mu\nu}(\rho)\, F_{i\nu}] \tag{64}$$

corresponds to Eq. (56), with j^R, $L_{i\mu, j\nu}$ as above, and

$$F_{i\nu} = -\partial_i U_\nu \tag{65}$$

2. Simple Fluid. The hydrodynamic equations of a simple fluid in Onsager form may be taken from the review of Schmitz.[8] We have $A = 5$ with

$$(\rho_0, \rho_1, \rho_2, \rho_3, \rho_4) = (\rho, \pi_1, \pi_2, \pi_3, \varepsilon) \tag{66}$$

where ρ is the mass density, π is the 3-vector momentum density, and ε is the energy density. The conjugate thermodynamic variables are related to the temperature T, chemical potential μ, and local velocity $\mathbf{v}$ by

$$\lambda_0 = \frac{\mu - v^2/2}{T}, \qquad \lambda_i = \frac{v_i}{T}, \qquad \lambda_4 = -\frac{1}{T} \tag{67}$$

The hydrodynamic equations in the Onsager form of Eq. (45) are specified by the reversible currents:

$$j^R_{i\mu} = v_i(\rho_\mu + p\delta_{4\mu}) + p\delta_{i\mu} \tag{68}$$

with p the (scalar) pressure, and by the Onsager coefficients given as

$$L_{i0,j\beta}=0 \tag{69}$$

$$L_{ik,jl}=\eta T(\delta_{ij}\delta_{kl}+\delta_{il}\delta_{kj})+(\zeta-\tfrac{2}{3}\eta)\,T\delta_{ik}\delta_{lj} \tag{70}$$

$$L_{ik,j4}=\eta T(v_i\delta_{kj}+v_k\delta_{ij})+(\zeta-\tfrac{2}{3}\eta)\,T\delta_{ik}v_j \tag{71}$$

$$L_{i4,j4}=(\kappa T+\eta v^2)\,T\delta_{ij}+(\zeta+\tfrac{1}{3}\eta)\,Tv_iv_j \tag{72}$$

subject to the symmetry condition

$$L_{i\alpha,j\beta}=L_{j\beta,i\alpha} \tag{73}$$

The quantities appearing in these equations are the transport coefficients: the shear viscosity η, the bulk viscosity ζ, and the thermal conductivity κ. In this situation, driving forces are usually introduced in the form of an "external" or "background" *stress tensor* τ and *heat current* $\mathbf{q}$ via

$$\tau_{ij}=L_{ij,k\beta}F_{k\beta} \tag{74}$$

$$-q_i+\tau_{ij}v^j=L_{i4,k\beta}F_{k\beta} \tag{75}$$

In this form, the driven equations take the form

$$\dot{\rho}=-\partial\cdot\boldsymbol{\pi} \tag{76}$$

$$\dot{\boldsymbol{\pi}}=-\partial\cdot[\mathbf{v}\boldsymbol{\pi}+p\mathbf{1}+\boldsymbol{\tau}'] \tag{77}$$

$$\dot{\varepsilon}=-\partial\cdot[\mathbf{v}(\varepsilon+p)+\mathbf{v}\cdot\boldsymbol{\tau}'+\mathbf{q}'] \tag{78}$$

with

$$\tau'_{ij}=-\eta(\partial_jv_i+\partial_iv_j)-(\zeta-\tfrac{2}{3}\eta)\,\delta_{ij}\partial\cdot\mathbf{v}+\tau_{ij} \tag{79}$$

and

$$q'_i=-\kappa\partial_iT+q_i \tag{80}$$

Equivalently, the driving forces may be introduced via $F_{i\mu}=\partial_iU_\mu$ and the "external potentials" given as

$$U_0=-\tfrac{1}{2}\beta u^2,\qquad U_i=\beta u_i,\qquad U_4=-\beta \tag{81}$$

in terms of "background" temperature $1/\beta$ and velocity $\mathbf{u}$. The latter are related to the "background" stress tensor and heat current via

$$\tau_{ij}=-\eta(\partial_iu_j+\partial_ju_i)-(\zeta-\tfrac{2}{3}\eta)\,\partial\cdot\mathbf{u}\delta_{ij} \tag{82}$$

and

$$q_i = -\kappa \partial_i (1/\beta) \tag{83}$$

In the above general hydrodynamic context we would like to formulate, as a conjecture, a generalization of the result of KOV. Of course, even to formulate a mathematically precise conjecture at this point is out of the question: on many points we must be very informal and inexact. Our starting point is a microscopic model, deterministic or stochastic, whose behavior on the macroscopic scale is determined by hydrodynamic laws of the sort discussed. We assume that the hydrodynamic laws may, in fact, be derived in the form of a hydrodynamic scaling limit as a separation-of-scales parameter ε tends to zero. For realistic equations, such as full Navier–Stokes equation of a simple fluid, this presumes a proper formulation of the multitime-scale limit, for which the following considerations might need to be modified. We assume, in any case, that the model provides us with a probability measure P^ε on the set of space-time hydrodynamic fields which is indexed by the ratio-of-scales parameter ε. (For a deterministic model, probability enters—only—from ignorance of the initial conditions.)

Now we first assume the analogue of Lemma 5.1 of KOV: namely, that for any ρ in a suitable class, there exists a *unique* U such that the driven hydrodynamic equations (56) hold with that ρ and U. Let us denote this U_μ by $U_\mu[\rho]$ and the corresponding force $F_{i\mu} = \partial_i U_\mu$ by $F_{i\mu}[\rho]$. Since ρ should always remain in the class satisfying the b.c. of (59), we may assume that

$$U_\mu \,|_{\partial \Lambda} = 0 \tag{84}$$

We now form the functional

$$I[\rho] = \frac{1}{4} \int_{-\infty}^{+\infty} d\tau \int_\Lambda dq \, L_{i\mu, j\nu}(\rho) \, F_{i\mu}[\rho] \, F_{j\nu}[\rho] \tag{85}$$

which has the physical interpretation of one-half the dissipation by the external forces $F_{i\mu}[\rho]$ required to produce the profile ρ. Observe that $I[\rho] \geqslant 0$ (possibly $= +\infty$). In terms of the functional I we can formulate the following conjecture.

Conjecture 1. Fluctuation-Dissipation Hypothesis. The measure P^ε has the large-deviations property with rate function $I[\rho]$ in the hydrodynamic limit as $\varepsilon \to 0$. In particular,

$$P^\varepsilon(\rho \in A) \sim \exp(-\varepsilon^{-d} \inf_{\rho \in A} I[\rho]) \tag{86}$$

for suitable sets A (e.g., I-continuous cylinder sets).

Note that the $\sim$ above has the same interpretation as in Eq. (17). The conjecture appears in a somewhat different form than the theorem of KOV: in particular, the rate function appears to have only a "dynamic" part and not to have the "static" contribution seen there. However, we shall see in the sequel that the rate function proposed here is in fact identical to that of KOV in their context. The correct prefactor of the rate function in the exponent is ε^{-d} in d dimensions, as may be seen from simple dimensional considerations (or comparison with the known static equilibrium result.) Our title for the conjecture, "Fluctuation-Dissipation Hypothesis," arises from the fact that the probability of a fluctuation is expressed directly in terms of the dissipation function. In fact, as we shall see shortly, the hypothesis gives a nonlinear generalization of the usual fluctuation-dissipation relation. (We refer here to the fluctuation-dissipation relation of the second kind, so-called. The fluctuation-dissipation relation of the first kind is an identity between fluctuation correlation functions and the imaginary or dissipative part of linear response functions. It does not concern us here.)

Let us consider whether the hypothesis is a reasonable one. In the first place, since the Onsager matrix is positive definite, $I[\rho]=0$ only for vanishing external forces, i.e., when ρ satisfies the hydrodynamic equations (45). Thus, the hypothesis implies that the hydrodynamic equations are obeyed with probability approaching one (law of large numbers) as $\varepsilon \to 0$. Now consider the case of linear fluctuations about a solution $\bar{\rho}(q, \tau)$ of the hydrodynamic equations.[14] Write

$$\rho(q, \tau) = \bar{\rho}(q, \tau) + \varepsilon^{d/2} \xi(q, \tau) \tag{87}$$

and introduce "random fluxes" $j_{i\mu}$ by the relation

$$L_{i\mu, j\nu}(\bar{\rho}(q, \tau))\, F_{j\nu}(q, \tau) = \varepsilon^{d/2} j_{i\mu}(q, \tau) \tag{88}$$

If we introduce these expressions into the driven hydrodynamic equations, we obtain to $O(\varepsilon^{d/2})$,

$$\dot{\xi}_\mu(q, \tau) = \hat{\mathscr{H}}_{\mu\nu}[\bar{\rho}; q, \tau]\, \xi_\nu(q, \tau) + \partial \cdot \mathbf{j}_\mu(q, \tau) \tag{89}$$

where $\hat{\mathscr{H}}_{\mu\nu}[\bar{\rho}]$ is the hydrodynamic operator obtained by linearizing Eq. (45) about the solution $\bar{\rho}$. From the Fluctuation-Dissipation Hypothesis we guess that the "random fluxes" have a Gaussian probability distribution

$$P[j] \sim \exp\left[-\frac{1}{4} \int_{-\infty}^{+\infty} d\tau \int_{\Lambda} dq (L^{-1})_{i\mu, j\nu} (\bar{\rho}(q, \tau))\, j_{i\mu}(q, \tau)\, j_{j\nu}(q, \tau) \right] \tag{90}$$

which implies in particular the covariance

$$\langle j_{i\mu}(q,\tau)\, j_{j\nu}(q',\tau')\rangle = 2L_{i\mu,j\nu}(\bar{\rho}(q,\tau))\, \delta(q-q')\, \delta(\tau-\tau') \tag{91}$$

Thus we recover (formally) the usual assumptions of linear fluctuating hydrodynamics, including the standard fluctuation-dissipation relation (91) (i.e., for the nonequilibrium case, the so-called extended local equilibrium hypothesis[8]). This gives our hypothesis some plausibility.

The connection we propose between the probability of fluctuations and the dissipation function strongly recalls the result of Onsager and Machlup discussed in the introduction. However, the dissipation function employed by OM was expressed, somewhat differently than ours above, as an "action functional." This analogy suggests we may be able to find here also an "Onsager–Machlup Lagrangian," a functional $L[\dot{\rho}, \rho]$ of the state variables ρ and their first time derivatives, so that

$$I[\rho] = \int_{-\infty}^{+\infty} d\tau\, L[\dot{\rho}, \rho] \tag{92}$$

In that case, the minimization of the functional $I[\rho]$ subject to appropriate constraints may be reduced to solving a set of Euler–Lagrange equations. In fact, the rate function in the present case may indeed be cast in such a form. For this purpose, we introduce the operator $\hat{G}_{\mu\nu}(\rho)$, defined as the integral operator with the kernel $G_{\mu\nu}(\rho; q, q')$ satisfying

$$\hat{A}_{\mu\sigma}(\rho)\, G_{\sigma\nu}(\rho; q, q') = \delta_{\mu\nu}\delta(q-q') \tag{93}$$

with Dirichlet boundary conditions:

$$G_{\sigma\nu}(\rho; q, q')\,|_{q,q'\in\partial\Lambda} = 0 \tag{94}$$

Observe that

$$\begin{aligned} I[\rho] &= \frac{1}{4}\int_{-\infty}^{+\infty} d\tau \int_{\Lambda} dq\, L_{i\mu,j\nu}(\rho)\, \partial_i U_\mu[\rho]\, \partial_j U_\nu[\rho] \\ &= -\frac{1}{4}\int_{-\infty}^{+\infty} d\tau \int_{\Lambda} dq\, U_\mu\, \partial_i[L_{i\mu,j\nu}(\rho)\, \partial_j U_\nu] \qquad \text{using } U_\mu|_{\partial\Lambda} = 0 \\ &= -\frac{1}{4}\int_{-\infty}^{+\infty} d\tau \int_{\Lambda} dq\, U_\mu(\hat{A}_{\mu\lambda} U_\lambda) \\ &= -\frac{1}{4}\int_{-\infty}^{+\infty} d\tau \int_{\Lambda} dq\, (\hat{G}_{\mu\nu}\hat{A}_{\nu\kappa} U_\kappa)(\hat{A}_{\mu\lambda} U_\lambda) \end{aligned} \tag{95}$$

or

$$I[\rho] = -\frac{1}{4}\int_{-\infty}^{+\infty} d\tau \int_{\Lambda} dq \int_{\Lambda} dq'\, G_{\mu\nu}(\rho; q, q')(\hat{A}_{\mu\lambda} U_{\lambda})(q)(\hat{A}_{\nu\kappa} U_{\kappa})(q') \tag{96}$$

Finally, using Eq. (56), i.e.,

$$(\hat{A}_{\mu\lambda} U_{\lambda})(q) = \dot{\rho}_{\mu}(q) + \partial_i j^R_{i\mu}(\rho; q) + \hat{A}_{\mu\lambda}(\rho) \frac{\delta S}{\delta \rho^{\lambda}(q)} [\rho] \tag{97}$$

gives

$$I[\rho] = \int_{-\infty}^{+\infty} d\tau\, L[\dot{\rho}, \rho] \tag{98}$$

with the "Onsager–Machlup Lagrangian"

$$\begin{aligned} L[\dot{\rho}, \rho] = -\frac{1}{4} \int_{\Lambda} dq \int_{\Lambda} dq'\, G_{\mu\nu}(\rho; q, q') & \\ \times \left\{ \dot{\rho}_{\mu}(q) + \partial_i j^R_{i\mu}(\rho; q) + \hat{A}_{\mu\lambda}(\rho) \frac{\delta S}{\delta \rho^{\lambda}(q)} [\rho] \right\} & \\ \times \left\{ \dot{\rho}_{\nu}(q') + \partial_j j^R_{j\nu}(\rho; q') + \hat{A}_{\nu\kappa}(\rho) \frac{\delta S}{\delta \rho^{\kappa}(q')} [\rho] \right\} & \end{aligned} \tag{99}$$

This is precisely the expression proposed by Graham in his 1978 Solvay Conference report.[9] For that reason, we shall refer to the above expression as the Onsager–Machlup–Graham Lagrangian.

Observe that $L[\dot{\rho}, \rho] \geqslant 0$ and $L[\dot{\rho}, \rho] = 0$ if and only if the hydrodynamic equations hold for ρ:

$$\dot{\rho}_{\mu} = -\partial_i j^R_{i\mu}(\rho) + \hat{A}_{\mu\nu}(\rho)\, \lambda_{\nu}(\rho) \tag{100}$$

Thus, we see again from this representation that the absolute minimum $I[\rho] = 0$ is achieved only if the hydrodynamic equations are satisfied. In particular, the solutions of Eq. (100) are also solutions of the Euler–Lagrange equations

$$\frac{d}{d\tau}\left(\frac{\delta L}{\delta \dot{\rho}_{\mu}(q)}\right) - \frac{\delta L}{\delta \rho_{\mu}(q)} = 0 \tag{101}$$

Notice, however, that the latter are *second order* in time. Therefore, there are solutions of the latter which are *not* solutions of the former. This is important in solving certain problems of minimization (e.g., analyzing the growth of a spontaneous fluctuation). In general, to find the most probable hydrodynamic history subject to a sequence of constraints such as $\rho(q, \tau_1) = \rho_1(q), \ldots, \rho(q, \tau_p) = \rho_p(q)$, one must solve a sequence of initial-final-value problems for the Euler–Lagrange equations (101) obtained from Eq. (99). This generalizes to the nonlinear domain *Onsager's principle of least dissipation*, which, in its original form, allowed the imposition of constraints only of $O(\varepsilon^{d/2})$ away from the expected evolution.

5. GENERALIZED THERMODYNAMIC POTENTIALS FOR THE STEADY STATE AND THE PRINCIPLE OF LEAST DISSIPATION

In large-deviations theory, there is a general principle known as the *contraction principle*. Simply stated, it says that, given a certain system of random quantities indexed by n and having the large-deviations property with rate function I as $n \to \infty$, then the rate function for a *reduced* system of quantities, requiring less information, is obtained by minimizing the rate function I relative to the superfluous information. More formally, we have the following result.

Theorem 3. Contraction Principle.[30] Let $\mathscr{X}$ and $\mathscr{Y}$ be Polish spaces with the σ-algebra of Borel sets and $\pi: \mathscr{X} \to \mathscr{Y}$ a continuous map of $\mathscr{X}$ onto $\mathscr{Y}$. Given a sequence of probability measures P_n on $\mathscr{X}$ which have the large-deviations property with rate function I as $n \to \infty$, then $Q_n = P_n \circ \pi^{-1}$ is a sequence of probability measures on $\mathscr{Y}$ having the large-deviations property with rate function J given by

$$J(y) = \inf_{x \in \pi^{-1}(y)} I(x) \tag{102}$$

This principle has many applications, e.g., to get the rate function for a random variable from the rate function for its probability distribution.

Here, we discuss how the contraction principle may be applied heuristically in the hydrodynamic context to obtain the rate function $K[\rho_0]$ for *static* fluctuations from the proposed rate function $I[\rho]$ for dynamic fluctuations, as

$$K[\rho_0] = \inf_{\rho : \rho(q,0) = \rho_0(q)} I[\rho] \tag{103}$$

In fact, with such a constraint at time $t = 0$, we may clearly write

$$K[\rho_0] = \inf_{\rho : \rho(q,0) = \rho_0(q)} \int_{-\infty}^{0} d\tau \, L[\dot{\rho}, \rho] \tag{104}$$

since the future time integral is minimized by the solution of the hydrodynamic equation with initial condition $\rho_0(q)$ and gives zero contribution. (We may note in passing that this is a nonlinear generalization of the *Onsager regression hypothesis*: given that a fluctuation has occurred at time zero, the most probable future course is that the fluctuation will decay away according to the hydrodynamic law.) In the equilibrium case, the static rate function $K[\rho_0]$ should be just the entropy or free energy, if our hypothesis is correct. For the nonequilibrium case, $K[\rho_0]$ would be an analogue of the free energy, appearing in a fluctuation formula for the steady state:

$$Q^{\varepsilon}(\rho_0 \in A) \sim \exp(-\varepsilon^{-d} \inf_{\rho_0 \in A} K[\rho_0]) \tag{105}$$

where Q^{ε} is the measure on the static density fields induced by the stationary measure for the microscopic dynamics. The quantity $K[\rho_0]$ represents physically the (minimum) total dissipation by the external field, integrated over the past, to produce the fluctuation ρ_0 at time zero. Clearly, if ρ_{ss} is any stationary solution of the hydrodynamic equations, then $K[\rho_{ss}] = 0$, which is the minimum value (even if ρ_{ss} is a nonequilibrium state with nonzero entropy production.) For this reason, we should think of $K[\rho_0]$ as the *(total) excess dissipation* by the external field required to produce ρ_0, above any internal dissipation in the solutions ρ_{ss} themselves. From this discussion it emerges that the stationary solutions of the hydrodynamic equations are, in fact, characterized by a variational principle, namely, a *principle of least excess dissipation*. This is a nonlinear, steady-state version (or a contraction to the steady state) of Onsager's principle of least dissipation.[7,31,32] Unlike Prigogine's principle of minimum entropy production,[6] which is valid only in the linear regime close to global equilibrium, the present principle is valid arbitrarily far from equilibrium. Of course, it is not clear that the principle is a more useful characterization of the steady-state profiles than the stationary solution condition itself.

In general, the only expression available for the excess dissipation function is that in Eq. (104), which involves an infinite-time integration and a minimization. However, Graham has found a set of conditions under which K may be evaluated exactly, which includes the equilibrium situation as a special case and yields there the appropriate free energy.[9] This supports the theory, at least in the equilibrium case. We shall now explain the conditions, following closely the exposition in ref. 9. Consider the case in which a *generalized potential* $\phi[\rho]$ and a *drift velocity* $v_\mu(\rho; q)$ can be defined, so that the hydrodynamic equation may be rewritten as

$$\dot{\rho}_\mu(q) = v_\mu(\rho; q) + \hat{A}_{\mu\nu}(\rho) \frac{\delta \phi}{\delta \rho^\nu(q)} \tag{106}$$

such that

$$\int_\Lambda dq\, v^\mu(\rho; q)\,\frac{\delta\phi}{\delta\rho^\mu(q)} = 0 \tag{107}$$

and

$$\left.\frac{\delta\phi}{\delta\rho^\mu(q)}\right|_{\partial\Lambda} = 0 \tag{108}$$

Of course, v^μ and ϕ are not independent: the requirement that the hydrodynamic equation take the form in Eq. (106) gives

$$v_\mu(\rho; q) = -\partial\cdot\mathbf{j}^R_\mu - \hat{A}_{\mu\nu}\,\frac{\delta}{\delta\rho^\nu}\,(\phi + S) \tag{109}$$

Note that the condition of Eq. (107) states that $\phi[\rho]$ is stationary under the evolution by v_μ.

If such a ϕ can be introduced, then it plays the role of a *Lyapunov variable* under the assumptions of a *unique* steady state and suitable regularity properties. Of course, the uniqueness assumption will not hold whenever the stationary solution bifurcates, as it does in many interesting cases: e.g., the Rayleigh–Bénard system at and above the threshold for convection. Even in that particular case, above the critical Rayleigh number for convective instability, but beneath that for a secondary instability, there is still a single stable stationary solution (corresponding to convective rolls). We shall confine ourselves in all the following discussion to the case where there is a unique stable stationary solution ρ_{ss}, and later in the paper discuss the modifications required for the case of multiple stationary solutions with open domains of attraction. Now, we have

$$\begin{aligned}
\frac{d}{d\tau}\phi[\rho] &= \int_\Lambda dq\,\dot{\rho}_\mu(q)\,\frac{\delta\phi}{\delta\rho^\mu(q)} \\
&= \int_\Lambda dq\left(\hat{A}_{\mu\nu}\frac{\delta\phi}{\delta\rho^\nu(q)}\right)\frac{\delta\phi}{\delta\rho^\mu(q)} \qquad \text{using Eqs. (106), (107)} \\
&= -\int_\Lambda dq\, L_{i\mu,j\nu}(\rho)\,\partial_i\left(\frac{\delta\phi}{\delta\rho^\mu(q)}\right)\partial_j\left(\frac{\delta\phi}{\delta\rho^\nu(q)}\right) \\
&\leqslant 0
\end{aligned} \tag{110}$$

In the steady state, $\dot{\phi}$ obviously vanishes, i.e., achieves its maximum. By positive definiteness of the Onsager coefficients, we can conclude that

$$\partial\left(\frac{\delta\phi}{\delta\rho^\mu(q)}\,[\rho_{ss}]\right) = 0 \tag{111}$$

Together with the boundary condition (108), this yields

$$\frac{\delta\phi}{\delta\rho^{\mu}(q)}[\rho_{ss}]=0 \tag{112}$$

Thus, ρ_{ss} is an extremal of ϕ. Let us assume good global convergence of arbitrary ρ_0 to ρ_{ss} under the hydrodynamic evolution for $\tau\to\infty$ and also continuity of $\phi[\rho]$. If $\rho_\tau(q)$ is the solution with initial condition ρ_0, then

$$\phi[\rho_{ss}]=\lim_{\tau\to+\infty}\phi[\rho_\tau]\leqslant\phi[\rho_0] \tag{113}$$

using $\dot{\phi}\leqslant 0$ for the latter. In this situation, therefore, ρ_{ss} is an absolute minimum of ϕ. Without loss of generality, we may take $\phi\geqslant 0$ and the minimum of ϕ to be at 0, i.e., $\phi[\rho_{ss}]=0$.

We now go back to our examples.

Examples

1. Purely Diffusive Systems. Consider a purely diffusive system with arbitrary nonequilibrium boundary conditions:

$$-\frac{\delta S}{\delta\rho^{\mu}(q)}=\lambda_{\mu}^{0}(q), \qquad q\in\partial\Lambda \tag{114}$$

In this case, we define

$$\phi[\rho]=-S[\rho]-\int_{\Lambda}dq\,\lambda_{ss}^{\mu}(q)\,\rho_{\mu}(q)+\Psi \tag{115}$$

with Ψ a normalization constant so that $\phi[\rho_{ss}]=0$. Here λ_{ss} is the steady-state profile of the conjugate chemical potentials, specified as the solution of

$$\partial\cdot[L_{\mu\nu}(\lambda_{ss}(q))\,\partial\lambda_{ss,\nu}(q)]=0 \tag{116}$$

with boundary condition

$$\lambda_{ss,\mu}(q)=\lambda_{\mu}^{0}(q), \qquad q\in\partial\Lambda \tag{117}$$

In this case, Eq. (109) yields

$$v_{\mu}(\rho;q)\equiv 0 \tag{118}$$

so that Eq. (107) is satisfied trivially. Also, Eq. (108) is easily checked to hold. The condition $\phi[\rho_{ss}] \leqslant \phi[\rho]$ is verified here without any assumption of global convergence to ρ_{ss}, so that ϕ is a Lyapounov variable by our earlier argument. Notice that if λ_μ^0 are strictly constant, then so are $\lambda_{ss,\mu}$, and ϕ is the standard thermodynamic potential.

2. Simple Fluid at Equilibrium. We impose here *constant* values only of the conjugate variables $\lambda_\mu = \lambda_\mu^0$, $\mu = 0, 1, 2, 3, 4$ at the boundary. We then take as ϕ the standard thermodynamic potential

$$\phi[\rho] = -S[\rho] - \lambda_\mu^0 N_\mu[\rho] + \Psi \tag{119}$$

with

$$N_\mu[\rho] = \int_\Lambda dq\, \rho_\mu(q) \tag{120}$$

It then follows from Eq. (66) that for this choice

$$v_\mu(\rho; q) = -\partial \cdot \mathbf{j}_\mu^R \tag{121}$$

It is again easy to verify by a simple calculation that Eq. (108) is satisfied

$$\left.\frac{\delta\phi}{\delta\rho^\mu(q)}\right|_{\partial\Lambda} = 0 \tag{122}$$

Furthermore, it is also well known that

$$\int_\Lambda dq\, v^\mu(\rho; q)\, \frac{\delta\phi}{\delta\rho^\mu(q)} = 0 \tag{123}$$

for this case (e.g., see ref. 33; in fact, it is a direct consequence of the microscopic Liouville theorem).

We now wish to show that, when such a "generalized potential" ϕ can be introduced, it is, in fact, the static rate function K introduced earlier. However, we must assume the additional *reversibility conditions*:

$$v_\mu(\tilde{\rho}; q) = -\varepsilon_\mu v_\mu(\rho; q) \tag{124}$$

$$\phi[\tilde{\rho}] = \phi[\rho] \tag{125}$$

It is easy to check that these hold in the examples above.

Proposition 1.[9] $K[\rho_0] \equiv \inf_{\rho : \rho(q,0) = \rho_0(q)} I[\rho] = \phi[\rho_0]$.

Formal Proof. Using the definitions of v_μ and ϕ, we may rewrite the Onsager–Machlup Lagrangian as

$$L[\dot{\rho}, \rho] = -\frac{1}{4}\int_\Lambda dq \int_\Lambda dq'\, G_{\mu\nu}(\rho; q, q') \times \left\{\dot{\rho}_\mu(q) - v_\mu(\rho; q) - \hat{A}_{\mu\lambda}(\rho)\frac{\delta\phi}{\delta\rho^\lambda(q)}[\rho]\right\} \times \left\{\dot{\rho}_\nu(q') - v_\nu(\rho; q') - \hat{A}_{\nu\kappa}(\rho)\frac{\delta\phi}{\delta\rho^\kappa(q')}[\rho]\right\} \tag{126}$$

We may now express this, according to tradition, as

$$L[\dot{\rho}, \rho] = \frac{1}{2}\left[\Phi(\dot{\rho} - v, \dot{\rho} - v) + \Psi\left(\frac{\delta\phi}{\delta\rho}, \frac{\delta\phi}{\delta\rho}\right) + \dot{\phi}\left(\dot{\rho}, \frac{\delta\phi}{\delta\rho}\right)\right] \tag{127}$$

where

$$\Phi(\dot{\rho} - v, \dot{\rho} - v) = -\frac{1}{2}\int_\Lambda dq \int_\Lambda dq'\, G_{\mu\nu}(\rho; q, q') \times [\dot{\rho}_\mu(q) - v_\mu(\rho; q)][\dot{\rho}_\nu(q') - v_\nu(\rho; q')] \tag{128}$$

$$\Psi\left(\frac{\delta\phi}{\delta\rho}, \frac{\delta\phi}{\delta\rho}\right) = -\frac{1}{2}\int_\Lambda dq \left[\hat{A}_{\mu\lambda}(\rho)\frac{\delta\phi}{\delta\rho^\lambda(q)}\right]\frac{\delta\phi}{\delta\rho^\mu(q)} \tag{129}$$

and

$$\dot{\phi}\left(\dot{\rho}, \frac{\delta\phi}{\delta\rho}\right) = \int_\Lambda dq\, \dot{\rho}^\mu(q)\frac{\delta\phi}{\delta\rho^\mu(q)} \tag{130}$$

Notice, by our reversibility conditions, that $L[\dot{\rho}, \rho]$ may be decomposed into parts even and odd under time reversal:

$$L_e[\dot{\rho}, \rho] = \frac{1}{2}\left[\Phi(\dot{\rho} - v, \dot{\rho} - v) + \Psi\left(\frac{\delta\phi}{\delta\rho}, \frac{\delta\phi}{\delta\rho}\right)\right] \tag{131}$$

$$L_o[\dot{\rho}, \rho] = \frac{1}{2}\int_\Lambda dq\, \dot{\rho}^\mu(q)\frac{\delta\phi}{\delta\rho^\mu(q)} \tag{132}$$

It is easy to see that

$$\frac{d}{d\tau}\left(\frac{\delta L_o}{\delta\dot{\rho}_\mu(q)}\right) - \frac{\delta L_o}{\delta\rho_\mu(q)} = 0 \tag{133}$$

identically, so that the Euler–Lagrange equations for L are equivalent to those for L_e:

$$\frac{d}{d\tau}\left(\frac{\delta L_e}{\delta\dot{\rho}_\mu(q)}\right) - \frac{\delta L_e}{\delta\rho_\mu(q)} = 0 \tag{134}$$

Alternatively, note that L_o is a total time derivative, which makes no contribution to the action. We therefore see that under our assumptions, the Euler–Lagrange equations for L are *time-reversal invariant.* In particular, not only are the solutions of the hydrodynamic equations,

$$\dot{\rho}_\mu(q) = v_\mu(\rho; q) + \hat{A}_{\mu\nu}(\rho)\frac{\delta\phi}{\delta\rho^\nu(q)} \tag{135}$$

solutions of the Euler–Lagrange equations, but also the solutions of the *anti*-hydrodynamic equations,

$$\dot{\rho}_\mu(q) = v_\mu(\rho; q) - \hat{A}^T_{\mu\nu}(\rho)\frac{\delta\phi}{\delta\rho^\nu(q)} \tag{136}$$

where $(\hat{A}^T_{\mu\nu} f_\nu)(q) = \partial_i(L_{j\nu, i\mu}\,\partial_j f_\nu)(q)$, satisfy the Euler–Lagrange equations.

We can easily see then that the unique minimizing profile—call it $\bar{\rho}$—subject to the constraint $\rho^\mu(q, 0) = \rho_0(q)$ is the solution for $\tau \in [0, +\infty)$ of the *initial-value problem*

$$\dot{\rho}_\mu(q, \tau) = v_\mu(\rho; q) + \hat{A}_{\mu\nu}(\rho)\frac{\delta\phi}{\delta\rho^\nu(q, \tau)}$$

$$\rho_\mu(q, 0) = \rho_{0\mu}(q)$$

and for $\tau \in (-\infty, 0]$ of the *final-value problem*

$$\dot{\rho}_\mu(q, \tau) = v_\mu(\rho; q) - \hat{A}^T_{\mu\nu}(\rho)\frac{\delta\phi}{\delta\rho^\nu(q, \tau)}$$

$$\rho_\mu(q, 0) = \rho_{0\mu}(q)$$

We note that $\lim_{\tau \to \pm\infty} \bar{\rho}(q, \tau) = \rho_{ss}(q)$ and recall that $\phi[\rho_{ss}] = 0$. Since $\langle \hat{A}_{\mu\nu}(\bar{\rho})(\delta\phi/\delta\rho^\nu)[\bar{\rho}], (\delta\phi/\delta\rho^\mu)[\bar{\rho}]\rangle = \langle \hat{A}^T_{\mu\nu}(\bar{\rho})(\delta\phi/\delta\rho^\nu)[\bar{\rho}], (\delta\phi/\delta\rho^\mu)[\bar{\rho}]\rangle$ and since also

$$\begin{aligned}
&\langle \hat{G}_{\mu\nu}(\bar{\rho})(\dot{\bar{\rho}}_\nu - v_\nu(\bar{\rho})), \dot{\bar{\rho}}_\mu - v_\mu(\bar{\rho})\rangle \\
&\quad = \langle \hat{G}^T_{\mu\nu}(\bar{\rho})(\dot{\bar{\rho}}_\nu - v_\nu(\bar{\rho})), \dot{\bar{\rho}}_\mu - v_\mu(\bar{\rho})\rangle \\
&\quad = \left\langle \hat{G}^T_{\mu\nu}(\bar{\rho})\,\hat{A}^T_{\nu\kappa}(\bar{\rho})\frac{\delta\phi}{\delta\rho^\kappa}[\bar{\rho}], \hat{A}^T_{\mu\lambda}(\bar{\rho})\frac{\delta\phi}{\delta\rho^\lambda}[\bar{\rho}]\right\rangle \qquad \text{by Eq. (136)} \\
&\quad = \left\langle \hat{A}^T_{\mu\lambda}(\bar{\rho})\frac{\delta\phi}{\delta\rho^\lambda}[\bar{\rho}], \frac{\delta\phi}{\delta\rho^\mu}[\bar{\rho}]\right\rangle \\
&\quad = -\left\langle \dot{\bar{\rho}}^\mu - v_\mu(\bar{\rho}), \frac{\delta\phi}{\delta\rho^\mu}[\bar{\rho}]\right\rangle \qquad \text{again by Eq. (136)} \\
&\quad = -\left\langle \dot{\bar{\rho}}^\mu, \frac{\delta\phi}{\delta\rho^\mu}[\bar{\rho}]\right\rangle \qquad \text{by Eq. (123),}
\end{aligned} \tag{137}$$

it follows that

$$
\begin{aligned}
I[\bar\rho] &= -\frac{1}{4}\int_{-\infty}^{0} d\tau \left[\langle \hat G_{\mu\nu}(\bar\rho)(\dot{\bar\rho}_\nu - v_\nu(\bar\rho)), \dot{\bar\rho}_\mu - v_\mu(\bar\rho)\rangle \right. \\
&\quad \left. + \left\langle \hat A_{\mu\nu}(\bar\rho)\frac{\delta\phi}{\delta\rho^\nu}[\bar\rho], \frac{\delta\phi}{\delta\rho^\mu}[\bar\rho]\right\rangle - 2\left\langle \dot{\bar\rho}^\mu, \frac{\delta\phi}{\delta\rho^\mu}[\bar\rho]\right\rangle \right] \\
&= \int_{-\infty}^{0} d\tau \int_\Lambda dq\, \dot{\bar\rho}^\mu(q)\frac{\delta\phi}{\delta\rho^\mu(q)}[\bar\rho] \\
&= \int_{-\infty}^{0} d\tau \frac{d}{d\tau}\phi[\bar\rho] \\
&= \phi[\rho_0] \quad \text{QED}
\end{aligned}
$$

This proposition was stated by Graham.[9] The basic idea of the proof goes back to Onsager and Machlup.[7] The proposition shows, incidentally, that the rate function proposed in our Fluctuation-Dissipation Hypothesis is identical to that of Kipnis, Olla, and Varadhan for their model. More generally, it is a consequence that the well-established equilibrium fluctuation theory of Einstein–Boltzmann follows from our hypothesis by contraction to a single time.

For the nonequilibrium steady state of purely diffusive models, the proposition implies that the "generalized potential"

$$\phi[\rho] = -S[\rho] - \int_\Lambda dq\, \lambda^\mu_{ss}(q)\, \rho_\mu(q) + \Psi \tag{138}$$

is the static rate function. It is important to observe that the static rate function for this case is the same as that for a local equilibrium distribution with the steady-state density profile (apparently not generally true according to the conjecture). In particular, if we consider linear fluctuations around the steady-state profile,

$$\rho(q) = \rho_{ss}(q) + \varepsilon^{d/2}\xi(q) \tag{139}$$

then a quadratic approximation to ϕ is

$$\phi^{(2)}[\xi] = \varepsilon^d \int_\Lambda dq\, \frac{(\xi(q))^2}{2\chi(\rho_{ss}(q))} \tag{140}$$

This leads one to expect that $\xi(q)$ is distributed in the steady state as a Gaussian random field with covariance

$$\langle \xi(q)\, \xi(q')\rangle_{ss} = \chi(\rho_{ss}(q))\, \delta(q-q') \tag{141}$$

On the other hand, the stationary covariance given by fluctuating hydrodynamics is

$$\langle \xi(q)\, \xi(q') \rangle_{ss} = \chi(\rho_{ss}(q))\, \delta(q-q') + C_{NE}(q, q') \tag{142}$$

where C_{NE} is a long-range part with power-law decay.[8,27] In fact, the latter covariance is proved to be the correct result in the hydrodynamic limit for the nonequilibrium steady state of the symmetric simple exclusion dynamics in contact with stochastic reservoirs.[27] This does not falsify our conjecture, but only implies that, if it is true, our expectation (141) is naive. Indeed, the results for $\varepsilon \to 0$ that

$$P^{\varepsilon}_{ss}(\rho \in A) \sim \exp(-\varepsilon^{-d} \inf_{\rho \in A} \phi[\rho]) \tag{143}$$

and

$$P^{\varepsilon}_{ss}\left(\xi = \frac{\rho - \rho_{ss}}{\varepsilon^{d/2}} \in B\right) \to \mu_C(B) \tag{144}$$

where μ_C is the Gaussian measure with covariance C as in Eq. (142), need not be in contradiction, as different quantities, ρ and ξ, respectively, are held fixed in the limit. It is further plausible that the limit with ρ fixed will miss the long-range correlations, as these are $O(\varepsilon^d)$ on the microscopic scale. It is rather remarkable that this apparent contradiction arises for the *static* rate function, which, after all, is calculable from our conjectured *dynamic* rate function that does give formally the same covariance for the random fluxes as in fluctuating hydrodynamics [cf. Eq. (91)]. Of course, even if the proposed static rate function is correct, the above observation shows that it contains less information about the static structure of the steady state than one might have hoped. In the following section we will adduce some further arguments in support of both the general conjecture and the static rate function resulting therefrom via contraction.

6. NONLINEAR LANDAU–LIFSHITZ FLUCTUATING HYDRODYNAMICS

The large-deviations theory we have presented above provides a framework for the calculation of fluctuation probabilities which incorporates consistently all of the hydrodynamic nonlinearities. However, other schemes have been proposed which may be, in fact, more familiar. In particular, nonlinear stochastic differential equations (SDEs) obtained by introducing Gaussian white noise as random fluxes into the full nonlinear hydrodynamic equations have often been proposed to represent the combined effect of hydrodynamic nonlinearities and molecular fluctuations (e.g., ref. 25, Chapter XVII). This scheme generalizes to the nonlinear case the linear fluctuating hydrodynamics of Landau and Lifshitz.[26] Particular applications are to study the influence of noise on the hydrodynamic instability at the critical Rayleigh number in a Rayleigh–Bénard cell,[34] to provide the basis for mode-coupling calculations of equilibrium time-correlation functions,[11] etc. In the present section, we review this approach to nonlinear hydrodynamic fluctuations, in particular to make comparison with the large-deviations method.

A very careful and lucid discussion of the traditional nonlinear fluctuating hydrodynamics (for a simple fluid at equilibrium) is contained in the paper of van Saarloos *et al.*[11] For the case of the simple fluid discussed there, the basic equations have the same form as our Eqs. (76)–(80):

$$\dot{\rho} = -\partial \cdot \boldsymbol{\pi} \tag{145}$$

$$\dot{\boldsymbol{\pi}} = -\partial \cdot [\mathbf{v}\boldsymbol{\pi} + p\mathbf{1} + \boldsymbol{\tau}'] \tag{146}$$

$$\dot{\varepsilon} = -\partial \cdot [\mathbf{v}(\varepsilon + p) + \mathbf{v} \cdot \boldsymbol{\tau}' + \mathbf{q}'] \tag{147}$$

with

$$\tau'_{ij} = -\eta(\partial_j v_i + \partial_i v_j) - (\zeta - \tfrac{2}{3}\eta)\,\delta_{ij}\partial \cdot \mathbf{v} + \tau_{ij} \tag{148}$$

and

$$q'_i = -\kappa\,\partial_i T + q_i \tag{149}$$

However, the assumption in the present context is that $\boldsymbol{\tau}$ and $\mathbf{q}$ are Gaussian generalized random fields with zero means and covariances

$$\begin{aligned}\langle \tau_{ij}(q, \tau)\,\tau_{kl}(q', \tau')\rangle = 2k_{\mathrm{B}}[L_\eta(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \tfrac{2}{3}\delta_{ij}\delta_{kl}) + L_\zeta\delta_{il}\delta_{kl}]\\ \times\ \delta(q - q')\,\delta(\tau - \tau')\end{aligned} \tag{150}$$

$$\langle \tau_{ij}(q, \tau)\,q_k(q', \tau')\rangle = 0 \tag{151}$$

and

$$\langle q_i(q, \tau)\,q_j(q', \tau')\rangle = 2k_{\mathrm{B}}L_q\delta_{ij}\delta(q - q')\,\delta(\tau - \tau') \tag{152}$$

The continuum equations above produce, in fact, divergences at short distances and require, for a proper definition, a lattice or high-momentum cutoff. In the present case, the discretized SDEs are equivalently interpreted either in the sense of Ito or Stratonovich.[11] A chief result of the analysis of van Saarloos *et al.* is that the above stochastic differential equations have as their stationary distribution the Einstein distribution:

$$P_{\mathrm{eq}}[\rho] \sim e^{S[\rho]/k_{\mathrm{B}}} \tag{153}$$

if and only if the following relations between transport and Onsager coefficients should hold:

$$\eta = \frac{L_\eta}{T(q, \tau)}, \qquad \zeta = \frac{L_\zeta}{T(q, \tau)}, \qquad \kappa = \frac{L_q}{T^2(q, \tau)} \tag{154}$$

Here, $T(q, \tau) = T(\rho(q, \tau), \pi(q, \tau), \varepsilon(q, \tau))$ is the fluctuating temperature defined through the equilibrium relation $T(\rho, \pi, \varepsilon)$ in terms of the basic density fields. Because the Onsager coefficients defined through the covariances are necessarily independent of the fluctuating variables, the relations in (154) imply an unreal dependence of the transport coefficients on the fluctuating temperature and density which is not seen in physical systems. (It is interesting to observe that these restrictive physical conditions are the same as those under which Prigogine's principle of minimum entropy production is valid, if the Onsager coefficients are assumed to be spacetime constants. See ref. 6, Chapter V.) The conclusion of van Saarloos *et al.* on this situation was: "This implies that the assumption of Gaussian white noise is, strictly speaking, not compatible with the physical phenomenological laws. The present scheme nevertheless is valid if the temperature fluctuations may be considered to be sufficiently small so that the dissipative currents (5.23) and (5.24) [our Eqs. (148)–(149)] can be linearized completely in the fluctuating fields." It is not clear from this statement whether nonlinear fluctuating hydrodynamics should be valid near an equilibrium critical point, for example, where fluctuations are divergent and yet where precisely mode-coupling calculations of dynamic critical phenomena are required. The theory is subject to a further criticism that the nonlinear hydrodynamic equations are *not* obeyed in an average sense, even though the random fluxes are taken to have zero mean. This point has been made in a general context for nonlinear stochastic processes in the book of van Kampen.[10] Therefore, it is not clear how the deterministic law emerges in such a theory. Below we suggest a more precise statement of the sense of validity of nonlinear fluctuating hydrodynamics.

We have already seen in the previous section that the problem exposed by van Saarloos *et al.* does not occur in the large-deviations approach, where the standard Einstein–Boltzmann static fluctuation theory is reproduced without any special, unrealistic assumptions on the transport coefficients. In fact, the formulation of nonlinear Onsager–Machlup theory as a large deviation from a *hydrodynamic scaling limit* suggests a certain reformulation of nonlinear fluctuating hydrodynamics from which, in fact, the proposed large-deviations result can be rederived. We explain this new point of view for the simpler case of an A-component, purely diffusive system: it is an easy exercise to work out the analogous SDE for the general hydrodynamic situation, but the situation is somewhat confused by the multiple-time-scale problem. We take as our stochastic hydrodynamic equation

$$\partial_\tau \rho_\mu(q, \tau) = \partial_q \cdot [D_{\mu\nu}(\rho(q, \tau)) \cdot \partial_q \rho_\nu(q, \tau)] + (-2\hat{A}_q(\rho))^{1/2}_{\mu\nu}\, \dot{w}_\nu(q, \tau) \quad (155)$$

where $\dot{w}(q, \tau)$ is a Gaussian white noise field with delta covariance:

$$\langle \dot{w}_\mu(q, \tau)\, \dot{w}_\nu(q', \tau') \rangle = \delta_{\mu\nu} \delta(q - q')\, \delta(\tau - \tau') \tag{156}$$

and $(-\hat{A}_q(\rho))^{1/2}$ is the unique positive square root of the operator $-\hat{A}_q(\rho)$ introduced in Eq. (54). Thus, the fluctuation term is given as a space-time white-noise field transformed by a spatially nonlocal, integral operator, chosen, as we argue below, to yield the correct asymptotic probabilities. It is crucial to observe, however, that this fluctuation term may be rewritten as the divergence of a local current

$$(-2\hat{A}_q(\rho))^{1/2}_{\mu\nu}\, \dot{w}_\nu(q, \tau) = \partial \cdot \tilde{\mathbf{j}}_\mu(\rho; q, \tau) \tag{157}$$

with

$$\tilde{\mathbf{j}}_\mu(\rho; q, \tau) = -\mathbf{L}_{\mu,\nu}(\rho(q, \tau))\, \partial_q \int_\Lambda dq'\, ((-2\hat{G})^{1/2}\, (\rho_\tau))_{\nu,\lambda}\, (q, q')\, \dot{w}_\lambda(q', \tau) \tag{158}$$

This is essential in order to preserve the local conservation laws of the hydrodynamic densities, which are even more inviolable principles of the stochastic hydrodynamic theory than the fluctuation formulas, since they are exact features of the microscopic dynamics. In the case where the Onsager coefficients depend only on the average densities, rather than the full fluctuating hydrodynamic densities, the noise term is easily seen to be stochastically equivalent to the traditional one, since one may identify

$$\partial \cdot \mathbf{j}_\mu(q, \tau) = (-2\hat{A}_q(\bar{\rho}))^{1/2}_{\mu\nu}\, \dot{w}_\nu(q, \tau) \tag{159}$$

as Gaussian processes with the same covariance [where $\mathbf{j}$ is as in Eq. (91)] and, in that case, the equations are of the type considered by Bedeaux *et al.* However, in general, the proposed SDE differs essentially from that discussed in ref. 11, since the operator acting on the noise term is a function of the fluctuating densities, i.e., the noise enters *multiplicatively*[10] into the equations. It may be worth remarking that the choice of the noise term in our equation is just another expression of the fluctuation-dissipation relation, since the operator coefficient of the fluctuation term is (the square root of) the analogue of the dissipative Onsager coefficient for the continuous systems considered (see ref. 6, Chapter VI, Section 4). As before, the equation must be spatially discretized and is interpreted in the Ito sense. We leave a more careful mathematical treatment of the system to later work.

As discussed previously, we consider the deterministic hydrodynamic equation as valid in the sense of a law of large numbers for a microscopic particle system as the separation-of-scales parameter $\varepsilon \to 0$. However, the stochastic equation (155) is no longer invariant under a diffusive rescaling $q \to \varepsilon^{-1}q$, $\tau \to \varepsilon^{-2}\tau$, but now a power of ε appears in the white noise term:

$$\partial_\tau \rho_\mu(q, \tau) = \partial_q \cdot [D_{\mu\nu}(\rho(q, \tau)) \cdot \partial_q \rho_\nu(q, \tau)] + \varepsilon^{d/2}(-2\hat{A}_q(\rho))^{1/2}_{\mu\nu} \dot{w}_\nu(q, \tau) \tag{160}$$

Therefore, we suggest that the *proper* form of the nonlinear SDE is that in Eq. (160) containing explicitly the factor of ε, whose solution should be interpreted in an asymptotic sense as $\varepsilon \to 0$. We observe that the ε factor is in some sense just a book-keeping device, since it may be transformed away by a mere change of units back to microscopic length and time scales rather than lengths and times measured in units of macroscopic variations. However, since the ε parameter is an actual, specifiable quantity (Knudsen number) in concrete situations and the regime of validity of the hydrodynamic description is apparently one in which that parameter is small, it seems most illuminating to write the SDE in the form in which it explicitly appears. Since $\varepsilon \sim 10^{-5}$ for realistic systems, asymptotic results for $\varepsilon \to 0$ provide some justification for the application of the equations. Note, in particular, that the absolute magnitude of fluctuations might be large, e.g., near a critical point, and the theory still be applicable: it is only ε which is required to be small.

We now wish to describe the asymptotic results we expect to hold for Eq. (158) in the limit $\varepsilon \to 0$. We do so on the basis of analogy with a still simpler system of only finitely many degrees of freedom of the form

$$\dot{X}^\varepsilon_t = b(X^\varepsilon_t) + \varepsilon^{1/2}\sigma(X^\varepsilon_t)\, \dot{w}_t \tag{161}$$

Here, the *drift coefficients* $b^i(x)$ and *diffusion coefficients* $a^{ij}(x) = \sum_{k=1}^r \sigma^i_k(x)\, \sigma^j_k(x)$, i, $j = 1,..., r$, are assumed bounded functions on $\mathbf{R}^r$, uniformly continuous in x, and the diffusion matrix uniformly nondegenerate: $\sum_{ij} a^{ij}(x)\, c_i c_j \geqslant \mu \sum_i c_i^2$, $\mu > 0$. The SDE is interpreted in the Ito sense and defines a family of strong Markov diffusion processes $(X^\varepsilon, P^\varepsilon_x)$ on $\mathbf{R}^r$ when solved with the initial conditions $X^\varepsilon_0 = x \in \mathbf{R}^r$. Intuitively, the diffusions result from the deterministic dynamical system on $\mathbf{R}^r$ defined by vector field $b(x)$ when subject to random perturbations with strength measured by ε. The asymptotics of these diffusion processes have been studied in the limit as $\varepsilon \to 0$ in the work of Freidlin and Wentzell.[15,16] [Some of the same results have been obtained formally by Graham and Tel[35] by applying the method of steepest descent to the path integral solution of the Fokker–Planck equation corresponding to Eq. (161).] The statistical results obtained by Freidlin and Wentzell are of several sorts. First, they prove a law of large numbers (ref. 15, Theorem 2.1.2): if $x_t(x)$ is the solution of the deterministic equation $\dot{x}_t = b(x_t)$ with initial condition $x_0 = x$, and the drift and diffusion coefficients satisfy some technical conditions, then

$$\lim_{\varepsilon \to 0} P_x^\varepsilon\Big(\sup_{t \in [0, T]} |X_t^\varepsilon - x_t(x)| > \delta \Big) = 0 \tag{162}$$

Also obtained are results of the central limit theorem type (a special case of ref. 15, Theorem 2.2.2): if $X_t^{(1)}$ is the Gaussian process obtained as the solution of the linear stochastic equation

$$\dot{X}_t^{(1)} = b'(x_t(x))\, X_t^{(1)} + \sigma(x_t(x))\, \dot{w}_t, \qquad X_0^{(1)} \tag{163}$$

then under some assumptions on the coefficients it is shown that

$$\sup_{t \in [0, T]} \left[E_x^\varepsilon \left(\left| \frac{X_t^\varepsilon - x_t(x)}{\varepsilon^{1/2}} - X_t^{(1)} \right|^2 \right) \right]^{1/2} = O(\varepsilon^{1/2}) \tag{164}$$

uniformly in the initial point x.

However, most important for our purposes are the large-deviations theorems. The content of Theorem 5.3.1 of ref. 15 is a dynamical large-deviations result. Let $\mathscr{X} = C([0, T]; \mathbf{R}^r)$ be the space of continuous functions on $[0, T]$ into $\mathbf{R}^r$, topologized with the uniform metric $\rho(\phi, \psi) = \sup_{t \in [0, T]} |\phi_t - \psi_t|$, $a(\varepsilon) = \varepsilon^{-1}$, and $I[\phi]$ be the functional defined by

$$I[\phi] = \frac{1}{2} \int_0^T \sum_{ij} a_{ij}(\phi_t)(\dot{\phi}_t^i - b^i(\phi_t))(\dot{\phi}_t^j - b^j(\phi_t))\, dt \tag{165}$$

for absolutely continuous ϕ and $I[\phi] = +\infty$ otherwise. [Here, $a_{ij} = (a^{-1})^{ij}$.] Then, Freidlin and Wentzell show that P_x^ε for $\varepsilon \to 0$ has the large-deviations property on $\mathscr{X}$ with the sequence $a(\varepsilon)$ and rate function $I[\phi]$, uniformly with respect to the initial point x. They also obtain static large-deviations results under conditions sufficient to guarantee the existence of a unique stationary measure μ^ε for the diffusion (ref. 15, Theorem 6.4.3). Roughly speaking, the following is shown (for precise details see ref. 15): Suppose the deterministic dynamical system for the vector field $b(x)$ has a finite number of *stable* attractors $\mathscr{A}_i$, $i = 1, \ldots, s$ (unstable attractors may be neglected entirely). For each $i = 1, \ldots, s$ form the function

$$K_i(x) = \inf_{\phi \in C_{\mathscr{A}_i, x}((-\infty, 0], \mathbf{R}^r)} \int_{-\infty}^0 dt\, L(\dot{\phi}, \phi) \tag{166}$$

Here, $L(\dot{\phi}, \phi)$ is the same "Onsager–Machlup Lagrangian" as appears in Eq. (165) and $C_{\mathscr{A}_i, x}((-\infty, 0], \mathbf{R}^r)$ is the class of continuous paths in $\mathbf{R}^r$

starting on $\mathscr{A}_i$ at $t=-\infty$ and ending at x at $t=0$. Furthermore, introduce constants $W(\mathscr{A}_i)$ for each $i=1,..., s$, which can be defined in terms of the limits

$$\lim_{\varepsilon\to 0} \varepsilon \log\left(\frac{\mu^{\varepsilon}(\mathscr{A}_{i,\delta})}{\mu^{\varepsilon}(\mathscr{A}_{j,\delta})}\right) = -(W(\mathscr{A}_i) - W(\mathscr{A}_j)) \tag{167}$$

with $\mathscr{A}_{i,\delta}$ a δ-neighborhood of the attractor $\mathscr{A}_i$ for all $\delta < \delta_0$ sufficiently small. Obviously the $W(\mathscr{A}_i)$ are only defined up to a common additive constant, which may be chosen so that $\min_i W(\mathscr{A}_i)=0$. Then, form the function

$$K(x) = \lim_{1\leqslant i\leqslant s} (K_i(x) + W(\mathscr{A}_i)) \tag{168}$$

The result of Freidlin and Wentzell is that the stationary measures μ^{ε} have the large-deviations property with sequence $a(\varepsilon)=\varepsilon^{-1}$ and rate function $K(x)$ as $\varepsilon\to 0$.

By analogy to these results, we can infer the asymptotics for the nonlinear stochastic hydrodynamic equation (160) above as $\varepsilon\to 0$. For the comparison, we take $i\to(q, i)$ and $\varepsilon\to\varepsilon^d$. First, we should have a law of large numbers of the type of Eq. (162). Note that this answers the objection of van Kampen, since the hydrodynamic law enters as the most probable history of the system (this is similar to van Kampen's own resolution of the problem in terms of the "Ω-expansion"[(10)]). We note as well that the hydrodynamic analogue of Eq. (163) is equivalent to the standard linear fluctuating hydrodynamic equation (89), because of the identification in Eq. (159), and we infer as a central limit theorem that the corresponding infinite-dimensional Ornstein–Uhlenbeck process should be the limit of $(\rho^{\varepsilon}-\bar{\rho})/\varepsilon^{d/2}$ for $\varepsilon\to 0$.

Further, observe that the large-deviations theory we had previously proposed as a generalization of the result of KOV follows also formally for the nonlinear SDE (144) by analogy with the result of Freidlin and Wentzell, at least for the case where the deterministic hydrodynamic equations have a single stable attractor [compare the Lagrangians in Eq. (99) and Eq. (165) and the static rate functions in Eq. (104) and Eq. (168)]. Although the framework of nonlinear fluctuating hydrodynamics is essentially phenomenological, the fact that our version leads formally to the same results as the conjecture provides the latter some theoretical support. Fluctuating hydrodynamics suggests also how that conjecture might have to be modified in the situations (e.g. turbulence) where there is more than one stable hydrodynamic attractor. We should say further that nonlinear fluctuating hydrodynamics also makes many more detailed asymptotic predictions, e.g., higher-order corrections to the central limit theorem and asymptotic results for probability densities (local large-deviations results). It remains to be seen what, if any, of this finer information is accurate for the hydrodynamic limit of realistic particle systems.

7. CONCLUDING DISCUSSION

The present paper has been speculative, pedagogical, and programmatic. Obviously, many outstanding problems remain to be answered. The problem of properly formulating the "multi-time-scale hydrodynamic scaling limit" has been emphasized by many other authors before. We think it also desirable to carry out a careful discussion of nonlinear fluctuating hydrodynamics, along the lines discussed in the last section, in order to have at least a consistent phenomenological framework. A most important problem for the proposals advanced in this paper is the proof of the large-deviations theory for the hydrodynamic limit of Spohn's model of a nonequilibrium steady state.[27] This is probably the simplest hydrodynamic model in which long-range correlations appear. It is therefore the ideal model to test our conjecture for the static rate function and to resolve the issue of disagreement with the static covariance of linear fluctuation theory.

We hope we have convinced the reader of the necessity of understanding the asymptotic significance of hydrodynamic laws in the limit as a separation-of-scales parameter $\varepsilon \to 0$, in order to avoid paradoxes that arise in the naive approach. We imagine that the basic fluctuation-dissipation hypothesis we have proposed has a rather general validity, at least in all those cases of nonequilibrium thermodynamics where the system is locally in equilibrium. For systems which do not have that property, e.g., conductors in applied electric fields, we imagine that a form of the hypothesis is still true, but the dissipation function can no longer be calculated from a local equilibrium assumption. The "nonequilibrium potential" here introduced should be useful in the study of critical properties at "nonequilibrium phase transitions." We have concentrated on steady-state problems, but the Onsager–Machlup approach should extend to other nonequilibrium situations (e.g., hydrodynamics with time-dependent boundary conditions.) In all these cases, the availability of a variational principle should be a powerful tool.

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